Applied Mathematical Sciences, Vol. 2, 2008, no. 58, 2875 - 2890

Valuation of Real Options Using the Minimal Entropy Martingale Measure

Cyrus Seera Ssebugenyi

Department of Mathematics
Faculty of Science, Makerere University
ssebugenyis@math.mak.ac.ug, ssebugenyis@yahoo.com

Abstract

In this article, the problem of real options valuation in multinomial trees is investigated. A concrete single real options value based on the minimal entropy martingale measure is provided. Using the MEMM to valuate options in multinomial lattices is an easy procedure which can easily be implemented by practitioners.

Mathematics Subject Classification: 91B28, 91B06, 90B50

Keywords: Real Options Analysis, Minimal Entropy Martingale Measure, Multinomial Lattices

1 Introduction

The traditional approach used to valuate projects is based on the discounted cash flow method where projected future cash flows are discounted at a rate which reflects the riskiness of the project. This gives the present value for the projected cash flows. For instance, let $x = (x_1, x_2, \ldots, x_N)$ be cash flows expected at the end of a one period project X. If $p_j \geq 0,\ j = 1, \ldots, N$ is the probability that cash flow x_j will occur, then, the present value of the project is

$$X_0 = \frac{p_1 x_1 + \ldots + p_N x_N}{1 + k} \tag{1}$$

where k is the appropriate cost of capital.

This method has been widely criticized for its failure to account for managerial flexibility in the lifetime of the project. Management of real world projects requires flexibility on the part of managers whenever they receive new information regarding progression of their projects.

It was therefore important to develop valuation models which are able to capture the value of managerial flexibility over the lifetime of the project. Real options analysis is one such methodology that has become very popular in the recent past.

According to Klimek [16], in real options analysis (ROA), one attempts to apply the successful methods in financial options theory such as the Black and Scholes' option pricing formula to the world of project management instead. This is possible because there are a lot of similarities between financial and real options. For example, real options can be classified as calls or puts and their exercise style can be classified as European or American type. Moreover, the gross present value of the expected cash flows does correspond to the current value of the stock and the uncertainty in the project value corresponds to the volatility of the stock. A detailed analogue between financial and real options can be found in [3, 9, 15] and many other sources.

There are also several significant differences between real and financial options which may prohibit the direct application of the Black and Scholes's [1] option pricing equation to the area of project management. Copeland and Tufano [4] state that real options are more complex than financial options and no one can expect to capture all the contingencies associated with them in a standard Black and Scholes's option pricing formula.

Fortunately, over the years researchers have developed option pricing models which are more appropriate for project management. One such alternative is the binomial models in [7]. In addition to being a perfect approximation to the Black and Scholes's option pricing formula, the binomial model allows for early exercises. Copeland and Tufano [4] note that every node of the binomial lattice is a decision node where managers can incorporate decisions that need to be made over the life of the project. To determine the value of such a decision, you may create a portfolio to replicate decision values at each node. The no-arbitrage principle ensures that the value of the replicating portfolio matches the project value. This is the basic principle underlying the CRR [7] binomial model and was used widely in [3].

The argument of replication assumes that the underlying asset is tradeable and this issue brings out another significant difference between financial and real options. For the latter, the underlying asset is not tradeable, therefore replication is not possible. In other words, markets are incomplete. One way to overcome this and still be able to use financial options theory to price real options is to use a surrogate asset. A surrogate asset or a twin security was defined in [14] as a tradeable asset whose price process is closely related to the price process of the non-tradeable underlying asset of the real option. Unfortunately, using a surrogate asset can lead to wrong conclusions as was noted in [14]. They showed that using surrogate assets may lead to arbitrary prices which are consistent with absence of arbitrage and risk-neutral valuation. Klimek [16], noted that this (negative) phenomenon arises from using surrogate assets whose information structure is somewhat independent from the information of the underlying it is supposed to replace. According to Klimek [16], a surrogate asset is appropriate if it will match the one it replaces in every state of nature.

The task of looking for the appropriate surrogate asset is difficult because it is practically impossible to find a financial security whose payoffs in every state of nature matches the value of the project in question. Copeland and Antikarov [3] recommended the use of present value of the project without flexibility as the appropriate twin security. They stated that the present value of the project without flexibility is the best unbiased estimate of the market value of the project. What is more correlated with the project than the project itself? This Marketed Asset Disclaimer (MAD) assumption completes the market and the replication argument similar to that in [7] can easily be implemented in a binomial setting. It was recently established in [8] that the MAD assumption is indeed plausible for real options valuation.

In this article, we use the MAD assumption and other considerations to valuate real options in mutinomial lattices, using the minimal entropy martingale measure.

The rest of this article is structured as follows. In Section, 2, we discuss that our methods are embedded in rational valuation systems introduced in [16] and also highlight some of the advantages of risk neutral valuation over traditional methods. In Section 3, we introduce the minimal entropy martingale measure on a finite probability space and demonstrate with practical examples how we can use it to valuate real options. Section 4 concludes.

2 Rational Valuation Systems

Klimek [16] introduced a more general framework for the valuation of real options. According to Klimek, if $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{0\leq t\leq T}, \mathbb{P})$ is a filtered probability space with sample space Ω, probability measure $\mathbb{P}$ and filtration $\{\mathcal{F}\}_{0\leq t\leq T}$ with $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_T \subset \mathcal{F}$, then, a rational valuation system on this filtered probability space is defined as a family $(\pi_s^t, CF_t)_{0\leq s\leq t\leq T}$ where,

- $CF_t = L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ for $t = 0, \ldots, T$, is the space of bounded cash flows.
- $\pi_s^t : CF_t \longrightarrow CF_s$ for $0 \leq s \leq t \leq T$ is a linear bounded operator representing a valuation projection. These operators map non-negative functions onto non-negative functions and satisfy the following consistency conditions:

$$\begin{aligned} &\pi_s^u = \pi_s^t \circ \pi_t^u && \text{if } 0 \leq s \leq t \leq u \leq T, \\ &\pi_t^t(CF_t) = CF_t && \text{for } t = 0, \ldots, T. \end{aligned}$$

Klimek's rational valuation system is a framework consisting of several valuation models ranging from adjustment of discount factors as in [8] to changing of underlying probability distributions. For instance, if $X \in CF_t$, then the present value rule can be written as

$$\pi_s^t(X) = \frac{1}{1 + r_{st}} E_{\mathbb{P}}[X|\mathcal{F}_s]$$

where r_{st} is the appropriate cost of capital for the period $[s, t]$.

The same project X can be valuated by changing the underlying probability $\mathbb{P}$ to a risk-neutral probability $\mathbb{Q}$. The rational valuation rule can be written as

$$\pi_s^t(X) = E_{\mathbb{Q}}\left[\frac{B_s X}{B_t} \mid \mathcal{F}_s\right]$$

where B is an adapted process representing a risk-free bond.

Thus, according to Klimek, the project can be valuated by changing the underlying probability structure to a risk neutral one or by making step by step adjustments in the discount factors, the net present value approach can be used. As illustrated with Example 2, these two approaches may not in general be equivalent. The claim that the net present value rule with rightly adjusted discount rates is appropriate for project valuation, was substantiated in [8] where a real option valuation formula was formulated based on the present value of the project without flexibility.

Step by step adjustment of discount factors is a time consuming procedure and may not be practically feasible. On the other hand, the same risk-neutral probabilities, once determined, can be used throughout the valuation process and we are inclined to use this alternative.

The risk-neutral probability approach to valuation of real options has been restricted to binomial models. This is partly because of their simplicity but also that for higher order lattices, markets are generally incomplete and there is not a unique solution. According to Boute *et al* [2], the best that can be done is to derive bounds for real option values and they recommend that exact solutions be investigated in further research. In this article, we provide a concrete single real options value based on the minimal entropy martingale measure. Through a number of worked examples, we show how the minimal entropy martingale measure can be used in very practical ways.

3 Project Valuation Using the Minimal Entropy Martingale Measure

Let us consider the development of the present value of the project without flexibility in a single period. If $S_0 > 0$ is the present value of the project without flexibility, then, with probability p_j, which is assumed to be strictly positive, the value moves to $a_j S_0,\ j = 1, \ldots, N,\ 2 < N < \infty;\ N \in \mathbb{N}$. We assume that $a_i > a_j$ if $i < j$ and $a_j > 0$ for all j. Let r be the single period risk-free rate, then, the project is viable if and only if

$$a_1 > 1 + r > a_N. \tag{2}$$

This condition corresponds to the no arbitrage condition in finance. If $1 + r > a_1$ then, no one would invest in the project because in every state of nature, the return on the project would be (with positive probability) less than the risk-free return. On the other hand, if $a_N > 1 + r$, then, many players would borrow at a risk-free rate to invest in the project as their returns on investment would exceed the risk-free return. Shortly, this opportunity would be evened out.

project's cost of capital. According to the net present value rule, the project is worth investing in if A risk neutral probability measure can be defined as a probability measure under which the expected return on the project is equal to the risk-free rate. In other words, a strictly positive probability measure $\mathbb{Q} = (q_1, \ldots, q_N)$ is said to be a risk-neutral probability measure if and only if

$$E_{\mathbb{Q}}(\frac{S_1}{S_0}) = 1 + r \ \Leftrightarrow \sum_{i=1}^{N} a_i q_i = 1 + r, \tag{3}$$

where S_1 is the value of the project at the end of the period. Since $N > 2$, we are operating in an incomplete market where there are many risk-neutral probability measures. We choose a specific one that is closest to the objective probability measure in the sense that it minimizes the entropic distance with respect to the prior probability measure. We will need the following definitions.

Definition 1
Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on a general probability space $(\Omega, \mathcal{F})$, where the terms carry their usual meanings. A probability measure $\mathbb{Q}$ is said to be absolutely continuous with respect to $\mathbb{P}$ if $\mathbb{P}(A) = 0$ implies that $\mathbb{Q}(A) = 0$ for all $A \in \mathcal{F}$. Notation is $\mathbb{Q} \ll \mathbb{P}$.

Two probability measures are said to be equivalent if each of them is absolutely continuous with respect to the other. The notation is $\mathbb{Q} \sim \mathbb{P}$. We will focus on a finite sample space where all positive probability measures are equivalent.

We define two sets $\mathcal{M}_e$ and $\mathcal{M}$ as follows:

$$\begin{aligned} \mathcal{M} &= \left\{ \mathbb{Q} = (q_1, \ldots, q_N) : \mathbb{Q} \geq 0, \sum_{i=1}^{N} q_i = 1, \sum_{i=1}^{N} q_i a_i = 1 + r \right\}, \\ \mathcal{M}_e &= \{ \mathbb{Q} \in \mathcal{M} : \mathbb{Q} > 0 \}. \end{aligned}$$

The notation $\mathbb{Q} > 0$ implies that $q_j > 0$ for all $1 \leq j \leq N$ and similarly $\mathbb{Q} \geq 0$ implies that $q_j \geq 0$ for all $1 \leq j \leq N$.

$\mathcal{M}$ is the set of martingale probability measures which are absolutely continuous with respect to the prior probability $\mathbb{P}$ and $\mathcal{M}_e$ is the set of equivalent martingale measures. By the Fundamental Theorem of Asset Pricing (see [6, 11, 12, 13]), the set $\mathcal{M}_e \neq \emptyset$, since we have assumed an arbitrage-free market. In the current setting however, we can deduce from the following proposition that the no arbitrage condition in (2) ensures that $\mathcal{M}_e$ (equivalently $\mathcal{M}$) is non-empty.

Proposition 1
Let $c_1, c_2, \ldots, c_n$ be n (ordered) real numbers such that c_1 is the smallest and c_n is the largest and let $c \in \mathbb{R}$. There exists a strictly positive probability measure $\mathbb{P} = (p_1, \ldots, p_n)$ such that $\sum_{k=1}^{n} p_k c_k = c$ if and only if $c_1 < c < c_n$.

Proof

First, let us suppose that $n = 2$ and $c_1 < c < c_2$. Then, $\mathbb{P} = (\frac{c_2 - c}{c_2 - c_1}, \frac{c - c_1}{c_2 - c_1})$ solves the problem. In this case the probability measure is uniquely determined by c_1, c and c_2.

Let us suppose that $n > 2$ and let $c_1, \ldots c_k \leq c < c_{k+1}, \ldots c_N$. Let

$$b_1 = \frac{c_1 + \ldots + c_k}{k}$$

and

$$b_2 = \frac{c_{k+1} + \ldots + c_N}{N - k}.$$

Then,

$$c = \frac{b_2 - c}{b_2 - b_1} b_1 + \frac{c - b_1}{b_2 - b_1} b_2.$$

If we let $\lambda_1 = \frac{b_2 - c}{b_2 - b_1}$ and $\lambda_2 = \frac{b_2 - c}{b_2 - b_1}$, then,

$$c \;=\; \frac{b_2 - c}{b_2 - b_1} b_1 + \frac{c - b_1}{b_2 - b_1} b_2 = \lambda_1 b_1 + \lambda_2 b_2 = \sum_{i=1}^{k} \frac{\lambda_1}{k} c_i + \sum_{i=k+1}^{N} \frac{\lambda_2}{N - k} c_i.$$

Therefore, we can let the required probability measure to be $\mathbb{P} = (p_1, \ldots, p_n)$ where

$$p_i = \begin{cases} \frac{\lambda_1}{k}, & i = 1, \ldots, k, \\ \\ \frac{\lambda_2}{N-k}, & i = k+1, \ldots, N. \end{cases}$$

On the other hand, if $c_1, c_2, \ldots, c_n$ are n (ordered) real numbers such that $c_1 = \min\{c_1, \ldots, c_n\}, c_N = \max\{c_1, \ldots, c_n\}$ and $\sum_{k=1}^{n} p_k c_k = c$, then,

$$c_1 = \sum_{i=1}^{n} p_i c_1 < \sum_{i=1}^{n} p_i c_i = c < \sum_{i=1}^{n} p_i c_n = c_N.$$

Definition 2 *Relative Entropy (Cover and Thomas [5])*
Let $\mathbb{Q}$ and $\mathbb{P}$ be probability measures on a finite probability space Ω. The relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined as

$$I(\mathbb{Q},\mathbb{P}) = \sum_{\omega\in\Omega} \mathbb{Q}(\omega)\ln\frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)}.$$

We understand throughout the paper that $0\ln(0) = 0$.

Basic properties of relative entropy are well known. For example,

$$0 \le I(\mathbb{Q},\mathbb{P}) \le \infty. \tag{4}$$

Relative entropy gives a measure of how different two probability distributions are. It is not a metric though.

Definition 3 *(MEMM, See for example, [10])*
The probability measure $\widehat{\mathbb{Q}} \in \mathcal{M}$ is called the minimal entropy martingale measure (MEMM) if it satisfies

$$I(\widehat{\mathbb{Q}},\mathbb{P}) = \min_{\mathbb{Q}\in\mathcal{M}} I(\mathbb{Q},\mathbb{P}). \tag{5}$$

For a single period-finite probability model, we can determine the MEMM using the method of Lagrangian multipliers. The basic optimization problem is given as follows:

$$\begin{cases} \min \sum_{i=1}^N q_i \ln(\frac{q_i}{p_i}) \\ \\ \text{s.t. } \sum q_i a_i = 1+r,\ \sum_{i=1}^N q_i = 1 \text{ and } q_i \ge 0\ \forall\ 1\le i\le N. \end{cases} \tag{6}$$

It so happens that problem (6) has a unique solution $\widehat{\mathbb{Q}} = (\widehat{q}_1,\ldots,\widehat{q}_N)$ given by

$$\widehat{q}_i = \frac{p_i e^{-\gamma a_i}}{\sum_{j=1}^N p_j e^{-\gamma a_j}},\ i = 1,2,\ldots N \tag{7}$$

provided that there exists a constant γ which satisfies the following equation.

$$\sum_{i=1}^N p_i a_i e^{-\gamma a_i} = (1+r)\sum_{i=1}^N p_i e^{-\gamma a_i}. \tag{8}$$

The following lemma due to Frittelli [10], links the existence and uniqueness of γ to the no arbitrage assumption.

Lemma 1
There are no arbitrage opportunities if and only if equation (8) has a unique solution.

Proof:

The following proof is a modification of what was given in [10].

Let $M(x) = \sum_{i=1}^{N} a_i q_i(x)$ where $q_j(x) = \dfrac{p_i.e^{-xa_i}}{\sum_{i=1}^{N} p_i e^{-xa_i}}$, $j = 1, \ldots, N$. Clearly, $M \in C^1(\mathbb{R})$ and is the mean of a random variable $\mathbf{a}$ whose probability distribution is given as

$$\mathbb{P}(\mathbf{a} = a_j) = q_j(x) = \frac{p_i e^{-xa_i}}{\sum_{j=1}^{N} p_j e^{-xa_j}},\ i = 1, 2, \ldots N.$$

To show that there exists a unique γ such that $M(\gamma) = 1 + r$, we write the function M as follows:

$$M(x) = \frac{p_1 a_1 e^{-a_1 x} + p_2 a_2 e^{-a_2 x} + \ldots + p_N a_N e^{-a_N x}}{p_1 e^{-a_1 x} + p_2 e^{-a_2 x} + \ldots + p_N e^{-a_N x}}. \tag{9}$$

Then,

$$\lim_{x \to +\infty} M(x) = a_N < 1 + r < a_1 = \lim_{x \to -\infty} M(x). \tag{10}$$

Therefore, by the Intermediate Value Theorem, we conclude that there exists a constant $\gamma \in \mathbb{R}$ such that $M(\gamma) = 1 + r$.

To show uniqueness, it is sufficient to show that $M'(x) < 0$ for all $x \in \mathbb{R}$.

$$M'(x) = \sum_{i=1}^{N} a_i \frac{dq_j(x)}{dx} = -\sum_{i=1}^{N} a_i^2 q_i(x) + \left(\sum_{i=1}^{N} a_i q_i(x) \right)^2 = -Var(\mathbf{a}) < 0. \tag{11}$$

For the converse, let us suppose that there exists a constant γ satisfying equation (8). Let

$$q_j^* = \frac{p_j e^{-\gamma a_j}}{\sum_{i=1}^{N} p_i e^{-\gamma a_i}},\ j = 1, 2, \ldots, N. \tag{12}$$

Then, $\mathbb{Q}^* = (q_1^*, q_2^*, \ldots, q_N^*)$ is a probability measure. It is also a martingale probability measure since

$$\sum_{j=1}^{N} a_j S_0 q_j^* = \frac{\sum_{j=1}^{N} p_j a_j e^{-\gamma a_j}}{\sum_{i=1}^{N} p_i e^{-\gamma a_i}} = (1 + r) S_0. \tag{13}$$

The conclusion follows from Proposition 1.

As a corollary, if $N = 2$, then $\widehat{\mathbb{Q}} = (\widehat{q}_1, \widehat{q}_2)$ where

$$\widehat{q}_1 = \frac{1 + r - a_2}{a_1 - a_2} \text{ and } \widehat{q}_2 = \frac{a_1 - 1 - r}{a_1 - a_2},$$

and $\mathbb{Q}$ is the unique equivalent martingale measure for binomial models (See [7]).

Let $\tilde{x}_j$, $j = 1, \ldots, N$ be the state j payoff of the project $\widetilde{X}$ with a real option and let $\overline{\pi}$ be the value from immediate exercise. Then, the minimal entropy value of the project with a real option is given as

$$\pi(\widetilde{X}) = \max \left\{ \frac{1}{1+r} \sum_{j=1}^{N} \tilde{x}_j \widehat{q}_j,\ \overline{\pi} \right\} \tag{14}$$

and the minimal entropy value of the real option is therefore given as

$$\pi(\widetilde{X}) - X_0. \tag{15}$$

Using the following example from [8], we illustrate the procedure for the evaluation of real options using the MEMM.

Example 1 *Abandonment Option*
An abandonment option can be defined as an option to close out an investment prior to the fulfillment of the original conditions for termination.

Let us consider a project [][pp.8]DDV06 X with three possible outcomes, \$1.2, \$1 or \$0.8 and respective probabilities 25%, 50% and \$25%. The risk-free interest rate is 5% and the cost of capital is 10%.

Using these values, the present value X_0 for the project (without flexibility) is found to be

$$X_0 = \frac{0.25 \times 1.2 + 0.5 \times 1 + 0.8 \times 0.25}{1.1} = \$0.9091.$$

We want to determine the value of an abandonment option with a payoff of \$1 exercisable only at the end of the period.

Solution.

Using a risk-neutral argument, we can derive bounds on the value $\pi(\tilde{X})$ of the project with options as follows:

$$\inf_{\mathbb{Q}\in\mathcal{M}} E_{\mathbb{Q}} \left[\frac{\widetilde{X}}{1+r} \right] \leq \pi(\widetilde{X}) \leq \sup_{\mathbb{Q}\in\mathcal{M}} E_{\mathbb{Q}} \left[\frac{\widetilde{X}}{1+r} \right]$$

where

$$\mathcal{M} = \left\{ \mathbb{Q} = \left(q, \frac{17}{22} - 2q, q + \frac{5}{22} \right) : 0 < q < \frac{17}{44} \right\}.$$

Thus, the option value is found to be between \$0.043 and \$0.117.

Given this information, the MEMM is found to be $\widehat{\mathbb{Q}} = (0.149, 0.474, 0.377)$ and the minimal entropy value $\pi(\widetilde{X})$ for the abandonment option is found to be

$$\pi(\widetilde{X}) = \frac{0.149 \times 1.2 + 0.474 \times 1 + 0.377 \times 1}{1.05} - X_0 = \$0.072.$$

The certainty-equivalent version[1] of the net present value formula in [8] gives an option value between \$0.063 and \$0.076.

The minimal entropy martingale measure is not the only martingale probability measure that can be used to derive an exact solution. Other martingale probability measures such as the minimal variance martingale measure can be used.

Let $\mu = \sum_{i=1}^{N} p_i a_i$ and $\sigma^2 = \sum_{i=1}^{N} p_i (a_i - \mu)^2$. The minimal variance martingale measure (MVMM) (as in [10]) is defined as $\widetilde{\mathbb{Q}} = (\widetilde{q}_1, \ldots, \widetilde{q}_N)$ where

$$\widetilde{q}_j = p_j \left(1 + \frac{\mu - r}{\sigma^2}(\mu - a_j)\right), \; j = 1, \ldots, N. \tag{16}$$

If valued using the MVMM, the value of the abandonment option in Example 1 is found to be \$0.069.

Remark 1 *We remark that the MEMM is more appropriate as a pricing measure because as seen in equation (7), it is always equivalent to the objective probability measure. On the other hand, the minimal variance martingale measure is not in general equivalent to the objective probability measure. For further discussions on this issue, see [10] and references given there.*

Example 2
As another example, let us consider an option to contract or shrink a project. This can be achieved by selling or subletting part of the production facilities to another company. When exercised at a strike price K, the project's present value is shrunk by a factor β. The single period minimal entropy value of the project with the contraction option is given as

$$\pi(\widetilde{X}) = \frac{1}{1+r} \sum_{j=1}^{N} \widehat{q}_j \max(x_j, \beta x_j + K), \tag{17}$$

if the option is exercisable only at the end of the period. In Example 1, let us suppose that the project can be contracted by 25% thereby saving \$0.28 in operating expenses. What is the value of the contraction option?

A risk-neutral approach would give an option value between \$0.039 and \$0.047 and the certainty-equivalent version of the net present value formula proposed in [8] gives an option value between \$0.074 and \$0.104. We used the minimal entropy martingale measure and obtained a value of \$0.042 for the option to contract. Or, the total minimal entropy value of the project with a contraction option was found to be \$0.951.

[1] Using average market returns of 12% with standard deviation of 20%, as in [8].

Example 3 *Compound Options*[2]
Copano, a chemical firm is considering a phased investment plant. It will cost \$60 million immediately for permits and preparations, which will take a year. At the end of year one, the firm can invest \$400 million to complete the design phase. Managers believe that once the design phase is over, the firm has a two year window during which it can make a final investment worth \$800 million needed to build the plant.

This is an example of what is commonly known as compound options or sequential options: A \$60 million investment now, creates the right to invest \$400 million one year later, which if exercised creates the right to invest \$800 million to purchase the plant.

Based on NPV calculations, the firm is assumed to be worth $\$1,000$ million today but future values are uncertain and are random in nature. The volatility of the project is assumed to be 18.23% per annum and the risk-free rate is assumed to be 8% per annum.

At the discount rate of 10.83%, if the firm decides to invest in the second year, the present value of costs in year two, will be \$1072.2 but if it decides to invest in the third year, the present value of costs in year three, will be \$1008.56. Therefore, according to the net present value rule, it is worthless investing in this project. However, investments at the end of year one, two or three are options and will be exercised if deemed worth.

We describe how the value of this project can be determined in both the context of binomial and trinomial models. In both cases, we use a risk-neutral approach instead of using a replicating portfolio approach as in [4]. For multinomial models we use the MEMM to derive an exact value for the project.

3.0.1 Determining the value of the project using binomial Models

Following Copeland and Antikarov[3], the present value of the project without flexibility will act as the underlying asset and the binomial event tree for the underlying asset is shown in Figure 1. If $S > 0$ is the value at the beginning of a period, then, with risk-neutral probability $q > 0$ the value at the end of the period will be $uS, u > 0$ and with risk-neutral probability $1 - q$ it will be $dS, d > 0$ where q, u and d are parameters in the CRR [7] binomial model[3]. Having constructed a binomial lattice for the value of te project with no option,

[2] The following example is adopted (with permission) from [4].

[3] The CRR model is generally understood to imply up probability $q = \frac{e^{r\Delta t} - d}{u-d}$ and jump amplitudes $u = \frac{1}{d} = \exp(\sigma \Delta t)$ where σ is the volatility in annualized terms. With the same jump amplitudes , they also derived an alternative up probability $q = \frac{1}{2}\left(1 + \frac{(r - \frac{\sigma^2}{2})\sqrt{\Delta t}}{\sigma}\right)$. Using this alternative parametrization, the value of the project is found to be $\$M11.74$. We note that these two models are equivalent

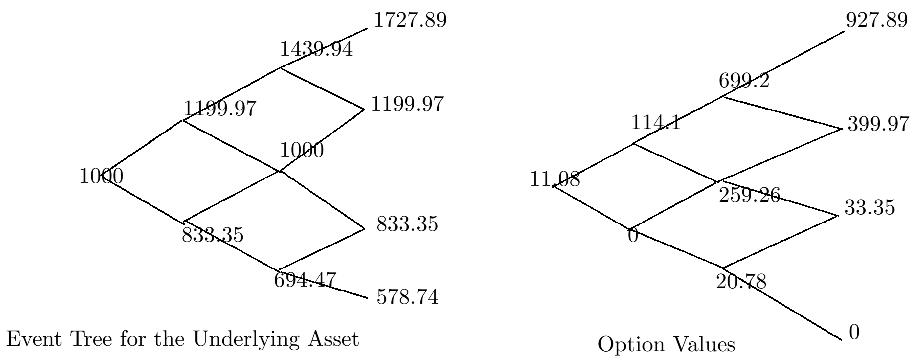

Figure 1: Copano's Binomial lattices showing the project and option values.

we then used single step risk neutral pricing relations to determine the value of compound options. The procedure is summarized in the following steps.

If $S_t^k, 0 \leq t \leq 3$ is the time t value of the project with no flexibility at node k of the binomial model, then

- At the end of the third year, the value of the project with a real option at node k is
$$C_3^k = \max\left\{S_3^k - 800, 0\right\},\ k = 1, \ldots, 4.$$

- At the end of the second year, the value of the project with a real option at node k is
$$C_2^k = \max\left\{S_2^k - 800, \frac{1}{1+r}(qC_3^k + (1-q)C_3^{k+1})\right\},\ k = 1, 2, 3.$$

- At the end of the first year, the value of the project with a real option at node k is
$$C_1^k = \max\left\{0, \frac{1}{1+r}(qC_2^k + (1-q)C_2^{k+1}) - 400\right\},\ k = 1, 2.$$

- At time zero, the value of the project with a real option is
$$C_0 = \max\left\{0, \frac{1}{1+r}(qC_2^1 + (1-q)C_2^2) - 60\right\} = \$11.08 \text{ million}$$

3.0.2 Trinomial models: Determining the value of the project using the minimal entropy martingale measure

To compute the minimal entropy value for Copano using trinomial lattices, we will need some additional information. In particular, we assume that present value $S > 0$ of the project without flexibility can move to uS with probability p_1, to S with probability p_2 and to dS with probability p_3. The corresponding jump amplitudes are assumed to be $u = \exp(\sigma\Delta t)$ and $d = \frac{1}{u}$.

To estimate the probabilities, we use equation (1). That is,

$$\frac{p_1 uS + p_2 S + p_3 dS}{1+k} = S.$$

Together with the condition that these probabilities add to one, p_1 and p_3 are computed as follows:

$$p_1 = \frac{(1+k-e^{-\sigma}) - p_2(1-e^{-\sigma})}{e^{\sigma} - e^{-\sigma}}$$

and

$$p_3 = 1 - p_2 - p_1,$$

where $0 \le p_2 \le 1$ is arbitrarily chosen in such a way that $0 \le p_1, p_3 \le 1$ and k is the appropriate discount factor. With these parameters the MEMM can be derived from equations (7) and (8).

We developed a trinomial event tree for the project without flexibility which is similar to the binomial event tree. We then used familiar steps as in the binomial model to determine the value of the project with a real option for varying values of p_2. Results are displayed in Table 1.

p_2	Project Value \$ million
0.00	11.08
0.05	8.41
0.10	5.71
0.15	2.95
0.20	0.12
≥ 0.25	0.00

Table 1: Real Option values using the MEMM.

Let us note that as $p_2 \to 0$, the minimal entropy value approaches the value computed using binomial models. In fact for $p_2 = 0$ the minimal entropy result coincides with the value computed using binomial models. We remark that trinomial models offer a more realistic development of the underlying asset. It is more natural to assume that the underlying asset can move up, move down or stay unchanged than to assume only up and down movements. However, for the more realistic trinomial models, markets are incomplete and as such, we use the minimal entropy martingale measure for purposes of option pricing.

The derivation of the minimal entropy martingale measure involves a constrained optimization of relative entropy when the necessary constraint is that the discounted value of the development of the present value is a martingale. The basic objective in this optimization problem is to derive a probability measure which correctly values the project without flexibility and is closest in the entropic distance to the prior probability measure. We point out that we do not have to be limited to the MAD assumption. A surrogate asset may still be used and it can be shown that using the current methods, the arbitrariness of prices as pointed out in [14], will not occur. Moreover, as a surrogate asset, we may use another project which is highly correlated with the one in question. All these securities are benchmark securities which can be included in the constraint equation in (6). Indeed, we can include as many independent constraints as we desire if we want to find a probability measure which correctly prices all these benchmark securities and is closest to the prior in the entropic distance. This procedure is commonly known as marking to market or model calibration (see for example, [17]).

4 Conclusion

The minimal entropy martingale measure was used to solve the problem of real options valuation in multinomial lattices. The MEMM yields a concrete single option value which is in some sense optimal. As illustrated by practical examples, the procedure is easy to implement and can be adopted by practitioners. Empirical research is necessary to determine how close minimal entropy prices are to actual values.

It was also shown that two approaches; certainty-equivalence and risk-neutral valuation can yield a range of option values which are non-overlapping. The relationship between these two approaches needs to be investigated in further research.

Lastly, I have discussed that with the current approach, the price process of any other marketed security (which is relevant to the pricing of the current project in question) can be used in the constraint equation. The chosen security could be the present value of the project without flexibility, the price process of an exchange traded asset or it could be the present value process of another project which shares similar features with the project in question.

Acknowledgments

This work is part of the ongoing Sandwich PhD project between the departments of Mathematics at Makerere University in Uganda and Uppsala University Sweden. This collaboration is financially supported by the International

Science programme Uppsala University Sweden. I thank Prof Maciej Klimek , Prof Patrick Mangheni and Dr John Mango for useful comments and support. I also thank the anonymous referee for very useful comments. Financial support from The Norwegian Programme for Development, Research and Education (NUFU) through collaboration with the Department of Mathematics, Makerere University and The African Mathematics Millennium Science Initiative (AMMSI) is acknowledged.

References

[1] F. Black and M. Scholes, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy,* **81** (1973), 637-654.

[2] R. Boute, and E. Demeulemeester and W. Herroelen, A Real Options Approach to Project Management, *International Journal of Production Research,* **42** (2004), 1715-1725.

[3] T. Copeland and V. Antikarov *Real Options,* Texere LLC, New York, 2001.

[4] T. Copeland and P. Tufano, A Real-World Way to Manage Real Options, *Harvard Business Review,* **82** (2004), 90-99.

[5] T. Cover and J.A. Thomas *Elements of Information Theory,* John Wiley, New York, 1991.

[6] J.C. Cox and S.S. Ross, The valuation of options for alternative stochastic processes, *Journal of Financial Economics,* **3** (1976), 145-166.

[7] J.C. Cox and S.S. Ross and M. Rubinstein, Option Pricing: A Simplified Approach, *Journal of Financial Economics,* **7** (1979), 229-263.

[8] B. De Reyck and Z. Degraeve and R. Vandenborre, Project Options Valuation with Net Present Value and Decision Tree Analysis, *European Journal of Operations Research,* (2006).

[9] A.K. Dixit and R.S. Pindyck, *Investment under Uncertainty,* Princeton University Press, 1994.

[10] M. Frittelli, Minimal Entropy Criterion for Pricing in One Period Incomplete Markets, *Working Paper. University of Brescia, Italy,* (1995).

[11] M. J. Harrison and M. D. Kreps, Martingales and Arbitrage in Multiperiod Securities Markets, *Journal of Economic Theory,* **20** (1979), 381-408.

[12] M. J. Harrison and S. R. Pliska, Martingales and Stochastic Integrals in the Theory of Continuous Trading, *Journal of Stochastic Processes and their Applications,* **81** (1981), 215-260.

[13] M. J. Harrison and S. R. Pliska, A Stochastic Calculus Model of Continuous Trading: Complete Markets, *Journal of Stochastic Processes and their Applications,* **15** (1983), 313-316.

[14] F. Hubalek and W. Schachermayer, The Limitations of No-Arbitrage Arguments for Real Options, *International Journal of Theoretical and Applied Finance,* **4** (2001), 361-373.

[15] J. Juniper, Extending Real Options Theory to Account for Property Investment Under Conditions of Uncertainty and Incomplete Markets, *Seventh Annual Pacific-Rim Real Estate Society Conference Adelaide South Australia,* (2001).

[16] M. Klimek, Rational Valuation Systems and Compatibility of Information, *Working Paper. Uppsala University Sweden,* (2005).

[17] L. Kruk, Limiting Distributions for Minimum Relative Entropy Calibration, *Journal of Applied Probability,* **41** (2004), 35-50.

Received: April 13, 2008